# Math 249 Monday Apr. 27

Jing operators

$$H_m^q = \langle z^m \rangle\, \Omega[zX]\, \Omega[(q-1)z^{-1}X]^\perp \qquad H_m^0 = B_m$$

$$= \langle z^m \rangle\, \underline{\Omega[zX]\,\Omega[-z^{-1}X]^\perp}\; \underline{\Omega[qz^{-1}X]^\perp} \qquad B_m s_\lambda = s_{(m;\lambda)}$$

$$= \sum_k q^k B_{m+k}\, h_k^\perp$$

Lemma $\displaystyle f^\perp s_\lambda = \sum_{w\in S_n} w\left(\frac{x^\lambda f(x_1^{-1},\ldots,x_n^{-1})}{\prod_{i<j}(1-x_j/x_i)}\right)_{pol}$ $\qquad x^\lambda f(x_1^{-1},\ldots,x_n^{-1})$

$\qquad x^{w(\mu+\rho)-\rho}$

Proof. Formula equiv. to

$$\langle s_\mu, f^\perp s_\lambda\rangle = \sum (-1)^{\ell(w)} \langle x^{w(\mu+\rho)-\rho}\rangle f(x^{-1})\, x^\lambda = \sum (-1)^{\ell(w)} \langle x^{w(\mu+\rho)-(\lambda+\rho)}\rangle f(x^{-1})$$

$$= \sum (-1)^{\ell(v)} \langle x^{-v(\lambda+\rho)+\mu+\rho}\rangle f(x^{-1})$$

$$= \sum (-1)^{\ell(v)} \langle x^{v(\lambda+\rho)-(\mu+\rho)}\rangle f(x)$$

$$= \sum (-1)^{\ell(v)} \langle x^{v(\lambda+\rho)-\rho}\rangle f(x)\, x^\mu$$

$$= \langle s_\lambda, f s_\mu\rangle \quad ✓$$

Computation:

$$H_m^q s_\lambda = \sum_k q^k \sum_{w\in S_n} w\left(\frac{x_1^{m+k}\, x_2^{\lambda_1}\cdots x_n^{\lambda_{n-1}}\, h_k(x_2^{-1},\ldots,x_n^{-1})}{\prod_{i<j}(1-x_j/x_i)}\right)_{pol}$$

$$= \sum_{w\in S_n} w\left(\frac{x_1^m\, x_2^{\lambda_1}\cdots x_n^{\lambda_{n-1}}}{\prod_{i<j}(1-x_j/x_i)\,\prod_{j\neq 1}(1-qx_1/x_j)}\right)_{pol}$$

$$= \sum_{w\in S_n/1\times S_{n-1}} w\left(\frac{x_1^m\, s_\lambda(x_2,\ldots,x_n)}{\prod_{j\neq 1}((1-x_j/x_1)(1-qx_1/x_j))}\right)_{pol}$$

$$\Rightarrow H_m^q f = \sum_{w\in S_n/1\times S_{n-1}} w\left(\frac{x_1^m\, f(x_2,\ldots,x_n)}{\prod_{j\neq 1}((1-x_j/x_1)(1-qx_1/x_j))}\right)_{pol}$$

$$\Rightarrow H_m^q H_\mu(x;q) = \sum_w w\left(\frac{x_1^m\, x_2^{\mu_1}\cdots x_n^{\mu_n}}{\prod_{i<j}((1-x_j/x_i)(1-qx_i/x_j))}\right)_{pol} = H_{(m;\mu)}(x;q)$$

$\Rightarrow$ Prop. $H_\mu(x;q) = H^q_{\mu_1} \cdots H^q_{\mu_\ell} \cdot 1$

$K_{\lambda\mu}(q) = \sum (-1)^{\ell(w)} \wp_q(w(\lambda+\rho)-(\mu+\rho))$

Ex. $H^0_m = B_m \Rightarrow H_\mu(x;0) = s_\mu$. $(q=0)$

$H^1_m = h_m \Rightarrow H_\mu(x;1) = h_\mu$ $(q=1)$ $\Longleftrightarrow K_{\lambda\mu}(1) = K_{\lambda\mu}$

(multiply by)

$e_i - e_j \quad i<j \quad (0\cdots 1 \cdots -1 \quad 0) \geq 0$

Triangularity (Upper) $H_\mu(x;q) = \sum_\lambda K_{\lambda\mu}(q)\, s_\lambda$

If $K_{\lambda\mu}(q) \neq 0$, some $\wp_q(w(\lambda+\rho)-(\mu+\rho)) \neq 0$

$\Rightarrow \lambda+\rho \geq w(\lambda+\rho) \geq \mu+\rho \Rightarrow \lambda \geq \mu$

(equality $\Rightarrow w=1$) $K_{\lambda\lambda}(q)=1$.

$H_\mu(x;q) = s_\mu + \sum_{\lambda>\mu} K_{\lambda\mu}(q)\, s_\lambda$

(Lower) Prop. $H_\mu[X(1-q);q] = \sum_{\lambda\leq\mu} b_{\lambda\mu}(q)\, s_\lambda$.

Lemma If $m \geq \lambda_1$, $e_k B_{m-k} s_\lambda = \sum_{\nu \leq (m;\lambda)} (?)\, s_\nu$

$s_{\cdots a\, b \cdots}$ $\quad a<b$

$a+1 \quad b$

$b \quad a+1$

$= - s_{\cdots b \mid a+1 \cdots}$

Proof: $B_{m-k} s_\lambda$

$= \pm s_{(\lambda_1-1, \dots, \lambda_j-1,\ m-k+j,\ \lambda_{j+1}, \dots, \lambda_\ell)}$

$(j+1)$'st entry

note that $m-k+j < \lambda_j \leq \lambda_1 \leq m$

$\Rightarrow j<k$ (if $j\neq 0$, also if $j=0$ wlog $k>0$)

$\geq$-maximal term of $e_k B_{m-k} s_\lambda$ is

$\pm s_{(\lambda_1, \dots, \lambda_j,\ m-k+j+1,\ \lambda_{j+1}+1, \dots, \lambda_k+1,\ \lambda_{k+1}, \dots)}$

Compare $i$th partial sum of with those of $(m, \lambda_1, \lambda_2, \dots)$

$i \leq j$ : $\lambda_1 + \cdots + \lambda_i \leq m + \lambda_1 + \cdots + \lambda_{i-1}$ $\qquad \lambda_i \leq \lambda_1 \leq m$

$j < i \leq k$ $\quad \lambda_1 + \cdots + \lambda_{i-1} + m-k+j+1 + i-(j+1) \leq m + \lambda_1 + \cdots + \lambda_{i-1}$ $\qquad i \leq k$

$i>k$ $\quad$ sums equal $\blacksquare$

Let $Q_\mu = H_\mu[X(1-q);q]$ $\qquad \Pi_{1-q}(f) = f[X(1-q)]$

$= \Pi_{1-q} H_\mu(x;q) = \Pi_{1-q} H^q_{\mu_1} \cdots H^q_{\mu_\ell} \cdot 1$ $\qquad \Pi_{1-q}^{-1} \cdot 1 = 1$

$= Q^q_{\mu_1} \cdots Q^q_{\mu_\ell}$ $\qquad Q^q_m \underset{def}{=} \Pi_{1-q} H^q_m \Pi_{1-q}^{-1}$

Operator

$\Pi_{1-q} f(x) \Pi_{1-q}^{-1} = f[X(1-q)]$

$\Pi_{1-q} \Omega[AX]^\perp \Pi_{1-q}^{-1} \cdot g(x) = g\left[\frac{X(1-q)+A}{1-q}\right] = g\left[X + \frac{A}{1-q}\right]$ $\qquad \Omega[AX]^\perp g = g[X+A]$

$\Pi_{1-q} \Omega[AX]^\perp \Pi_{1-q}^{-1} = \Omega(AX/(1-q))^\perp$ $\qquad = \Omega\left[\frac{A}{1-q} X\right]^\perp g$

$Q^q_m = \langle z^m \rangle \Pi_{1-q} \Omega[zX]\Omega[(q-1)z^{-1}X]^\perp \Pi_{1-q}^{-1} = \langle z^m \rangle \Omega[z(1-q)X] \Omega[-z^{-1}X]^\perp$

$= \langle z^m \rangle \Omega[-qzX] \; \Omega[zX]\Omega[-z^{-1}X]^\perp$

$= \sum_k (-1)^k q^k e_k B_{m-k}$

By lemma: If $m \geq \lambda_1$, $Q^q_m s_\lambda$ has only terms $s_\nu$ with $\nu \leq (m;\lambda)$

$\Rightarrow Q_\mu = H_\mu[X(1-q);q] = Q^q_{\mu_1} \cdots Q^q_{\mu_\ell} \cdot 1 = \sum_{\lambda \leq \mu} b_{\lambda\mu}(q) s_\lambda$ (by induction on $\ell$).

**Orthogonality:** Define $\langle f,g \rangle_q \underset{def}{=} \langle f, g[X(1-q)] \rangle = \langle g,f \rangle_q$

Prop. $\langle H_\lambda, H_\mu \rangle_q = 0$ if $\lambda \neq \mu$. $\qquad \langle \Omega[AX], \Omega[BX(1-q)] \rangle = \Omega[AB(1-q)]$ $\circlearrowleft$ $A \leftrightarrow B$

Pf. $\langle H_\lambda, H_\mu \rangle_q \neq 0 \Rightarrow \langle H_\lambda, H_\mu[X(1-q)] \rangle \neq 0$

$\Rightarrow$ Some $s_\nu$ occurs with coeff. $\neq 0$ in both $H_\lambda$, $H_\mu[X(1-q)]$

$\Rightarrow \lambda \leq \nu \leq \mu \Rightarrow \lambda \leq \mu$. Also $\mu \leq \lambda$ by symmetry. So $\lambda = \mu$. ∎

Characterization:

Any two of
- (i) $\langle H_\lambda, H_\mu \rangle_q = 0$ for $\lambda \neq \mu$
- (ii) $H_\mu = s_\mu + \sum_{\lambda > \mu} K_{\lambda\mu}(q) s_\lambda$ (for some $K_{\lambda\mu}(q)$).
- (iii) $H_\mu(X(1-q);q] = \sum_{\lambda \leq \mu} (?) s_\lambda$ and $\langle s_\mu \rangle H_\mu = 1$

**Ex.** By triangularity, $H_{1^n}[X(1-q);q] = (\text{scalar}) \cdot e_n \swarrow S_{(1^n)}$ $\qquad H_n(x;q) = H^q_n \cdot 1 = h_n$

i.e. $H_{1^n}(x;q) = (?)\, e_n[X/(1-q)]$ $\qquad (?)^{-1} = \langle e_n, e_n[X/(1-q)]\rangle$

$$= \langle h_n, h_n[X/(1-q)]\rangle$$

$$= h_n[1/1-q]$$

$$= h_n(1, q, q^2, \dots)$$

$$= \langle t^n \rangle \Omega[t(1+q+q^2+\dots)] = \langle t^n \rangle \prod_{i=0}^{\infty} \frac{1}{1-tq^i}$$

$\lambda \qquad q^{|\lambda|} \qquad t^{\ell(\lambda)}$ including 0's

$$= \text{OGF for } \lambda\text{'s s.t. } \ell(\lambda) \le n, \text{ by } q^{|\lambda|}$$

$$= \prod_{i=1}^{n} \frac{1}{1-q^i}$$

$$h_n[XY] = \sum_{|\lambda|=n} s_\lambda(x) s_\lambda(y)$$

$$H_{1^n}(x;q) = (1-q)(1-q^2)\cdots(1-q^n)\, e_n[X/(1-q)]$$

$$\omega H_{1^n}(x;q) = (1-q)(1-q^2)\cdots(1-q^n)\, h_n[X/(1-q)] = \sum_\lambda \underbrace{(1-q)\cdots(1-q^n)\, s_\lambda\left[\tfrac{1}{1-q}\right]}_{f_\lambda(q) = K_{\lambda^*, 1^n}(q)} s_\lambda(x)$$

$$f_\lambda(1) = |SYT(\lambda^*)| = |SYT(\lambda)|$$

$$f_\lambda(q) = \sum_{T \in SYT(\lambda)} q^{cc(T)}$$

**Ex.** $n=4$

| | |
|---|---|
| $q^6$ | $+\, s_{(1,1,1,1)}$ |
| $q^5$ | $+\, s_{(2,1,1)}$ |
| $q^4$ | $s_{(2,2)} + s_{(2,1,1)}$ |
| $q^3$ | $s_{(3,1)} + s_{(2,1,1)}$ |
| $q^2$ | $s_{(3,1)} + s_{(2,2)}$ |
| $q^1$ | $s_{(3,1)}$ |
| $q^0$ | $s_{(4)}$ |

Graded Frobenius Characteristic of $\mathbb{C}[x_1,\dots,x_n]/(e_1,\dots,e_n)$

$$\dim_{\mathbb{C}}\left( \mathbb{C}[x_1,\dots,x_n] \big/ (e_1,\dots,e_n) \right) = n!$$

$S_n \circlearrowright$

$$s_{(4)} + 3s_{(3,1)} + 2s_{(2,2)} + 3s_{(2,1,1)} + s_{(1,1,1,1)}$$

$n=4$

$x_1, \dots, x_4 / (x_1 + \dots + x_3)$ $\qquad \mathbb{C}^4/\mathbb{C}$

$s_{(3,1)}$

$s_{(2,2)}$

| | |
|---|---|
| 3 4 | 2 4 |
| 1 2 | 1 3 |
| 2 | 1, 3 |
| 2 | 4 |